ON CERTAIN IDENTITIES IN THETA FUNCTIONS[*]

BY

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1. The purpose of this paper is to obtain certain identities in Jacobi's theta functions by a new method. This method may be outlined as follows. We consider two sets of elements S and S_1, conceptually distinct, such that the elements of S are in one-to-one correspondence with the elements of S_1. In certain cases these elements are algebraic numbers and in other cases they are quaternions; in every case, corresponding elements have the same norms. We construct two functions $\Phi(x, y, \cdots)$ and $\Psi(x_1, y_1, \cdots)$, the terms of which are in one-to-one correspondence with the elements of S and S_1 respectively. Hence there is a one-to-one correspondence between the terms of Φ and those of Ψ. We then show that if $x_1, y_1, \cdots$ are properly defined linear functions of $x, y, \cdots$, the corresponding terms are equal and therefore $\Phi(x, y, \cdots) = \Psi(x_1, y_1, \cdots)$.

We shall prove Jacobi's well known fundamental identity,[†] i.e.

$$\Phi(x, y, z, w) \equiv \vartheta_3(x)\vartheta_3(y)\vartheta_3(z)\vartheta_3(w) + \vartheta_2(x)\vartheta_2(y)\vartheta_2(z)\vartheta_2(w) = \Phi(x_1, y_1, z_1, w_1),$$

where x_1, y_1, z_1, w_1 are certain linear functions of x, y, z, w. It will be shown that the terms of Φ are in one-to-one correspondence with the set of all (Hurwitz) integral quaternions.[‡] The above identity is then proved by employing certain well known properties of these quaternions. A generalization of this identity is obtained in a similar manner by employing sets of integral elements in generalized quaternion algebras. These sets were determined by the writer in a former paper,[§] employing a definition of integral element due to Dickson.[¶]

The other identities are obtained by employing sets of elements not integral in general. We also obtain Schroeter's identity by the same method.

The use of algebraic numbers and quaternions, and in particular their arithmetics, to obtain identities in theta functions is believed to be new and not without interest.

2. **Proof of Jacobi's fundamental identity.** Part of the argument used to establish this identity may be repeated verbatim later on to prove a generalization.

We shall recall certain well known definitions and properties of real quaternions. If $X = x + yi + zj + wk$ is a real quaternion, where i, j, k are the usual quaternion units, the conjugate of X is $X' = 2x - X$ and the norm of X is $N(X) = XX' = x^2 + y^2 + z^2 + w^2$. Then $N(XY) = N(X)N(Y)$ and $(XY)' = Y'X'$.

In the algebra of rational quaternions, the (Hurwitz) integral quaternions are those in the form

$$\tfrac{1}{2}(\xi + \eta i + \zeta j + \lambda k)$$

where ξ, η, ζ, λ are integers which are all even or all odd. The set S of all such quaternions is closed under multiplication.

By definition*

$$\vartheta_2(x, q) = \sum q^{(2m+1)^2/4} e^{(2m+1)xi},$$
$$\vartheta_3(x, q) = \sum q^{m^2} e^{2mxi},$$

where q is a complex number, $|q| < 1$. Since these functions are absolutely convergent we have, dropping the parameter q,

$$\vartheta_j(x)\vartheta_j(y)\vartheta_j(z)\vartheta_j(w) = \sum q^{(\xi^2 + \eta^2 + \zeta^2 + \lambda^2)/4} e^{(\xi x + \eta y + \zeta z + \lambda w)i},$$

where for $j = 2$ $[j = 3]$ the summation extends over all systems of four odd [even] integers ξ, η, ζ, λ. Assume for the present that x, y, z, w are real and let

$$\Omega = \tfrac{1}{2}(\xi + \eta i + \zeta j + \lambda k), \quad X = x + yi + zj + wk.$$

Let the real part of the quaternion Y be designated by $R(Y)$. Then

$$\Phi(x, y, z, w) = \sum_{\Omega} q^{N(\Omega)} e^{2iR(\Omega X')},$$

where Φ is as defined in §1 and the summation extends, without duplication or omission, over all elements of S. The terms of Φ are thus in one-to-one correspondence with the elements of S.

Let U be an element of S of norm unity and let x_1, y_1, z_1, w_1 be defined in terms of x, y, z, w by

$$X_1 = XU,$$

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* It will be understood throughout this paper that the limits of summation are $-\infty$ and $+\infty$ unless otherwise stated.

where

$$X_1 = x_1 + y_1 i + z_1 j + w_1 k.$$

Let Ω be an arbitrarily chosen element of S and let τ be the corresponding term of $\Phi(x, y, z, w)$. Since S is closed under multiplication,

$$\Omega U = \Omega_1$$

belongs to S. Let τ_1 be the term of $\Phi(x_1, y_1, z_1, w_1)$ which corresponds to Ω_1. We have

$$N(\Omega_1) = N(\Omega)N(U) = N(\Omega),$$

$$\Omega_1 X_1' = (\Omega U)(XU)' = (\Omega U)(U'X') = \Omega X'.$$

Hence

$$\tau = q^{N(\Omega)}e^{2iR(\Omega X')} = q^{N(\Omega_1)}e^{2iR(\Omega_1 X_1')} = \tau_1.$$

Since $\tau = \tau_1$ is a formal identity in x, y, z, w, we may discard our assumption that they are real. U belongs to S and therefore the same is true of U'. Hence by means of $\Omega U = \Omega_1$, we have a one-to-one correspondence between the terms of $\Phi(x, y, z, w)$ and $\Phi(x_1, y_1, z_1, w_1)$. And, as we have just seen, corresponding terms are equal. Therefore

(1) $$\Phi(x, y, z, w) = \Phi(x_1, y_1, z_1, w_1).$$

If we take as U one of the elements

$$\tfrac{1}{2}(\pm 1 \pm i \pm j \pm k)$$

and change the signs of certain of the variables, (1) becomes Jacobi's identity. The only other elements in S of norm unity are ± 1, $\pm i$, $\pm j$, $\pm k$. If we take one of these as U, (1) is trivial.

3. **The sets of integral elements in generalized quaternion algebras.** Consider the rational algebra A, with the basal numbers

$$1, \quad I = (\beta\gamma)^{1/2}i, \quad J = (\gamma\alpha)^{1/2}j, \quad K = (\alpha\beta)^{1/2}k,$$

where α, β, γ are positive odd integers such that $\alpha\beta\gamma$ contains no square factor >1. A is a subalgebra of the algebra of real quaternions and hence has the properties mentioned in §2.*

Two of α, β, γ are congruent, modulo 4, and therefore, after applying a cyclic substitution if necessary, we may assume that $\alpha \equiv \beta \pmod 4$. An integral element of A is defined as an element which belongs to a set S which

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* In J we considered an algebra A_1, with a multiplication table which involved certain parameters α and β. Let $\alpha = \alpha_1\delta, \beta = \beta_1\delta$, where δ is the g.c.d. of α and β, and assume that $\alpha < 0$ and $\beta > 0$. If in the multiplication table of A we replace $\alpha_1, \beta_1, \delta, K$ by $-\beta, \alpha, \gamma, \gamma K$ respectively, it will be seen that A_1 is equivalent to A. We shall quote the results of J in terms of the new notation.

has certain properties R, C, U, M.* By property C, if X, Y are elements of S then XY also belongs to S. We shall not employ explicitly the properties R, U, M.

Let $\alpha = A\alpha'$, $\beta = B\beta'$, $\gamma = C\gamma'$ where A, B, C are defined as follows. A is the product of all the prime factors A_i of α such that $-\beta\gamma$ is a quadratic non-residue of every A_i; or $A = 1$ if α contains no such factor. B and C are similarly defined with a cyclic substitution on A, B, C and on α, β, γ. If Q is the number of prime factors in $\alpha' \beta' \gamma'$, there are exactly $2^Q = t$ odd integers $H_i(i = 1, 2, \cdots, t)$, incongruent modulo $\alpha'\beta'\gamma'$ such that

(2) $\quad \beta\gamma H_i^2 + 1 \equiv 0 \ (\mathrm{mod}\ \alpha'), \ \gamma\alpha H_i^2 + 1 \equiv 0 \ (\mathrm{mod}\ \beta'), \ \alpha\beta H_i^2 + 1 \equiv 0 \ (\mathrm{mod}\ \gamma').$

Let H be one of the H_i and let $S_i[S_i', S_i'']$ be the totality of elements in the form

$$(3) \qquad \Omega = \frac{1}{2}\left(\xi + \frac{\eta}{\beta'\gamma'} I + \frac{\zeta}{\gamma'\alpha'} J + \frac{\lambda}{\alpha'\beta'} K\right),$$

where $\xi, \eta, \zeta, \lambda$ are rational integers such that

$$
\begin{aligned}
\zeta &\equiv B\gamma' H\lambda &&(\mathrm{mod}\ \alpha'),\\
(4)\qquad \eta &\equiv A\gamma' H\lambda &&(\mathrm{mod}\ \beta'),\\
\lambda &\equiv A\beta' H\zeta &&(\mathrm{mod}\ \gamma'),
\end{aligned}
$$

and such that the first [second, third] of the following conditions is satisfied:

$$
\begin{aligned}
\xi &\equiv \eta \equiv \zeta \equiv \lambda,\\
\xi &\equiv \eta, \quad \zeta \equiv \lambda,\\
\xi &\equiv \zeta, \quad \eta \equiv \lambda &&(\mathrm{mod}\ 2).
\end{aligned}
$$

We shall say that an element in the form (3) has the coördinates $\xi, \eta, \zeta, \lambda$. It will be noted that S_i is the intersection of S_i' and S_i''.

If $\alpha \equiv \gamma \ (\mathrm{mod}\ 4)$, by the proof of Theorem 3, J, S_i has the properties R, C, U, M. On the other hand, if $\alpha \equiv \gamma + 2 \ (\mathrm{mod}\ 4)$, the sets S_i' and S_i'' have these properties. For every H_i we thus have either one or two sets of integral elements.

From (2) it follows that (4) is equivalent to

$$
\begin{aligned}
\lambda &\equiv -\beta' CH\zeta &&(\mathrm{mod}\ \alpha'),\\
(5)\qquad \lambda &\equiv -\alpha' CH\eta &&(\mathrm{mod}\ \beta'),\\
\zeta &\equiv -\alpha' BH\eta &&(\mathrm{mod}\ \gamma').
\end{aligned}
$$

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* J, p. 57.

If η, ζ, λ satisfy (4) and if $\zeta \equiv \lambda$ or $\eta \equiv \lambda$ (mod 2), by employing (5) it may be shown that they may be written in the corresponding forms below:

(a) $\zeta \equiv \lambda$ (mod 2):
$$\eta = 2\beta'\gamma'm_1 + r,$$
$$\zeta = 2\gamma'\alpha'm_2 - \alpha'BHr + \beta\gamma'Hp,$$
$$\lambda = 2\alpha'\beta'm_3 - \alpha'CHr + \beta'p,$$
$$0 \leqq p < 2\alpha', \ 0 \leqq r < 2\beta'\gamma';$$

(6)

(b) $\eta \equiv \lambda$ (mod 2):
$$\eta = 2\beta'\gamma'm_1 + A\beta'Hr + \alpha\gamma'Hp,$$
$$\zeta = 2\gamma'\alpha'm_2 + r,$$
$$\lambda = 2\alpha'\beta'm_3 - \beta'CHr + \alpha'p,$$
$$0 \leqq p < 2\beta', \ 0 \leqq r < 2\alpha'\gamma',$$

where m_1, m_2, m_3, r, p are integers, r and p subject to the indicated inequalities. Conversely, if η, ζ, λ may be written in the forms (6a) [(6b)], they satisfy (4) and $\zeta \equiv \lambda$ [$\eta \equiv \lambda$] (mod 2).

4. **Construction of functions corresponding to the sets S_i, S_i', S_i''.** Let q be a complex number $|q| < 1$; $a = A/\alpha'$, $b = B/\beta'$, $c = C/\gamma'$, and

$$F_1(x, r) = \sum_{m_1} q^{bc\eta^2/4} e^{bc\eta xi},$$
$$G_1(x; r, p) = \sum_{m_2} q^{ca\zeta^2/4} e^{ca\zeta xi},$$
$$H_1(x; r, p) = \sum_{m_3} q^{ab\lambda^2/4} e^{ab\lambda xi},$$

where r, p are integers and η, ζ, λ are given by (6a). Let $F_2(x; r, p)$, $G_2(x, r)$, $H_2(x; r, p)$ be defined in the same way except that we use the expressions (6b) for η, ζ, λ. We may write

$$F_1(x, r) = q^{bcr^2/4} e^{bcrxi} \vartheta_3[BC(x - rki), q^{\beta\gamma}],$$

where $k = (\log q)/2$. The other five functions may be similarly expressed in terms of theta functions. Hence each of these six functions is absolutely convergent. Then

$$\Psi_0(s, t) = \vartheta_3(x) F_1(y, 2s) G_1(z; 2s, 2t) H_1(w; 2s, 2t)$$
$$= \sum q^{(\xi^2 + bc\eta^2 + ca\zeta^2 + ab\lambda^2)} e^{(\xi x + bc\eta y + ca\zeta z + ab\lambda w)i},$$

where the summation extends over all systems of four even integers ξ, η, ζ, λ such that η, ζ, λ may be written in the forms (6a) with r, p replaced by $2s$, $2t$ respectively. For every such system, there is a corresponding element of S_i,

$$\Omega = \frac{1}{2}\left(\xi + \frac{\eta}{\beta'\gamma'}I + \frac{\zeta}{\gamma'\alpha'}J + \frac{\lambda}{\alpha'\beta'}K\right).$$

Conversely, if Ω is an element of S_i with even coördinates, ξ, η, ζ, λ, there is a uniquely determined pair of integers s, t, $0 \leqq s < \beta'\gamma'$, $0 \leqq t < \alpha'$, such that $\Psi_0(s, t)$ contains exactly one term corresponding to Ω. Let

$$\Psi_1(s, t) \equiv \vartheta_2(x)F_1(y, 2s + 1)G_1(z; 2s + 1, 2t)H_1(w; 2s + 1, 2t).$$

The above argument regarding Ψ_0 may be repeated for Ψ_1 except that ξ, η, ζ, λ are odd, r is replaced by $2s+1$, and Ω has odd coördinates. Hence

(7)
$$\Phi_1(x, y, z, w) \equiv \sum_{s=0}^{\beta'\gamma'-1} \sum_{t=0}^{\alpha'-1} (\Psi_0(s, t) + \Psi_1(s, t))$$

$$= \sum_{\Omega} q^{N(\Omega)}e^{2R(\Omega X')i},$$

where we assume for the present that x, y, z, w are real,

$$X = x + \frac{y}{\beta'\gamma'}I + \frac{z}{\gamma'\alpha'}J + \frac{w}{\alpha'\beta'}K,$$

and the summation extends, without duplication or omission, over all elements of S_i. The terms of Φ_1 are thus in one-to-one correspondence with the elements of S_i.

We shall next construct functions Φ_2 and Φ_3, the terms of which are in one-to-one correspondence with the elements of S_i' and S_i'' respectively. As noted before, both S_i' and S_i'' contain S_i. Accordingly, we shall obtain Φ_2 and Φ_3 by adding properly chosen functions to Φ_1.

The η, ζ, λ coördinates of every element of S_i' may be written in the forms (6a) and $\xi \equiv \eta$, $\zeta \equiv \lambda$ (mod 2). Furthermore, if such an element does not belong to S_i, $\eta \equiv \lambda+1$ (mod 2), and therefore in (6a) $r \equiv p+1$ (mod 2). Therefore if we set

$$\phi_{20} \equiv \vartheta_3(x) \sum_{s=0}^{\beta'\gamma'-1} \sum_{t=0}^{\alpha'-1} F_1(y, 2s)G_1(z; 2s, 2t + 1)H_1(w; 2s, 2t + 1),$$

$$\phi_{21} \equiv \vartheta_2(x) \sum_{s=0}^{\beta'\gamma'-1} \sum_{t=0}^{\alpha'-1} F_1(y, 2s+1)G_1(z; 2s+1, 2t+1)H_1(w; 2s+1, 2t+1),$$

it follows that $\phi_{20}+\phi_{21}$ is equal to the expression on the right of (7), where in this case the summation extends over all the elements of S_i' which do not belong to S_i. Then

$$\Phi_2(x, y, z, w) \equiv \Phi_1 + \phi_{20} + \phi_{21} = \sum_{\Omega} q^{N(\Omega)}e^{2iR(\Omega X')},$$

where the summation extends over all the elements of S_i'. If we let

$$\phi_{30} \equiv \vartheta_3(x) \sum_{s=0}^{\gamma'\alpha'-1} \sum_{t=0}^{\beta'-1} F_2(y; 2s, 2t+1) G_2(z, 2s) H_2(w; 2s, 2t+1),$$

$$\phi_{31} \equiv \vartheta_2(x) \sum_{s=0}^{\gamma'\alpha'-1} \sum_{t=0}^{\beta'-1} F_2(y; 2s+1, 2t+1) G_2(z, 2s+1) H_2(w; 2s+1, 2t+1),$$

it may be shown as for Φ_2 that

$$\Phi_3(x, y, z, w) \equiv \Phi_1 + \phi_{30} + \phi_{31} = \sum_{\Omega} q^{N(\Omega)} e^{2iR(\Omega X')},$$

where now the summation extends over all the elements of S_i''.

5. Identities in functions which correspond to sets of integral elements. For each of the sets S_i, S_i', S_i'' we have a corresponding function $\Phi_j (j=1, 2, 3)$. As noted in §3, S_i has the properties R, C, U, M if $\alpha \equiv \gamma$ (mod 4), while S' and S'' have these properties if $\alpha \equiv \gamma+2$ (mod 4). Let S be one of the sets which has these properties and let Φ be the corresponding function. Let

$$X_1 = x_1 + \frac{y_1}{\beta'\gamma'} I + \frac{z_1}{\gamma'\alpha'} J + \frac{w_1}{\alpha'\beta'} K.$$

The argument of §2 beginning "Let U be an element of $S \cdots$" and leading to (1) may now be repeated word for word, it being understood that we employ the Φ, X_1 and S of this paragraph and the X and Ω of the preceding paragraph. We have then

THEOREM 1. *Let S be one of the sets S_i, S_i', S_i'' which has the properties R, C, U, M, and let Φ be the corresponding Φ_j. If U is an element of S of norm unity and if x_1, y_1, z_1, w_1 are defined in terms of x, y, z, w by means of $XU = X_1$, then*[*]

$$\Phi(x, y, z, w) = \Phi(x_1, y_1, z_1, w_1).$$

6. Special cases of Theorem 1. If α, β, γ are assigned numerical values, to write out explicitly the identity of the above theorem it is necessary to determine an element U as there defined, i.e. a unit belonging to S. In some cases a unit yields a trivial identity, as was seen at the end of §2. Furthermore, the algebra may have no unit which yields a non-trivial identity. This is the case if $\alpha = \beta = 1$, $\gamma = 7$; the only units being ± 1, $\pm K$.

By the proofs of Theorems 2 and 3 of J, every element with integral coördinates and integral norm belongs to a set of integral elements. The

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[*] We have considered only those algebras in which α, β, γ are odd. The same method is being used in a Kentucky thesis, employing the algebras in which one of these parameters is even and the results of Professor Darkow on the arithmetics of such algebras. See Annals of Mathematics, vol. 28, pp. 263–70.

problem of determining those algebras which contain units yielding non-trivial identities is therefore equivalent to that of determining those values of α, β, γ such that the equation

$$\alpha'\beta'\gamma'\xi^2 + \alpha'BC\eta^2 + \beta'CA\zeta^2 + \gamma'AB\lambda^2 = 4\alpha'\beta'\gamma'$$

has a solution in integers, two of which are $\neq 0$. This, of course, is a very difficult problem. We shall give two classes of algebras in which such a unit may be determined and the corresponding identity written out explicitly.

Let $\beta = \gamma = 1$ and let α be a product of distinct primes, each in the form $4n+1$. Then $A = B = C = \beta' = \gamma' = 1$, $\alpha = \alpha' = u^2 + v^2$, where u, v are relatively prime. We may take

$$U = \frac{1}{2}\left(1 + I + \frac{u+v}{\alpha}J + \frac{u-v}{\alpha}K\right).$$

For example, if $\alpha = 5$, $u = 2$, $v = 1$, $H = 3$, it will be found that Φ may be written

$$\Phi(x, y, z, w) = \vartheta_3(x, q)\vartheta_3(y, q)\sum_{t=0}^{4}q^{2t^2}e^{2(3z+w)ti/5}\vartheta_3(z - 6tki, q^5)\vartheta_3(w - 2tki, q^5)$$

$$+ \vartheta_2(x, q)\vartheta_2(y, q)\sum_{t=0}^{4}q^{2t^2}e^{2(3z+w)ti/5}\vartheta_2(z - 6tki, q^5)\vartheta_2(w - 2tki, q^5).$$

By the theorem,

$$\Phi(x, y, z, w) = \Phi(x_1, y_1, z_1, w_1),$$

where we find

$$x_1 = \frac{1}{2}\left(x - y - \frac{3z+w}{5}\right), \quad z_1 = \frac{1}{2}(3x - y + z + w),$$

$$y_1 = \frac{1}{2}\left(x + y + \frac{z-3w}{5}\right), \quad w_1 = \frac{1}{2}(x + 3y - z + w).$$

Similar results may be obtained if α is a product of distinct primes, each in the form $3n+1$, one of β, γ is unity and the other is 3, noting that the notation must be so chosen that $\alpha \equiv \beta$ (mod 4).

If the method of the preceding paragraphs be applied to the algebraic integers in the field defined by an imaginary cube root of unity and if we set

$$\Phi(x, y) \equiv \vartheta_3(x, q)\vartheta_3(y, q^3) + \vartheta_2(x, q)\vartheta_2(y, q^3),$$

we obtain the known identity

$$\Phi(x, y) = \Phi(x_1, y_1),$$

where

$$x_1 = \tfrac{1}{2}(x + y), \quad y_1 = \tfrac{1}{2}(3x - y).$$

7. Proof of Schroeter's identity. The sets of elements used hereafter will not be integral in general. Let r, p be positive integers and let s, t be integers such that t is prime to

$$\Delta \equiv rs^2 + pt^2.$$

Let $\rho = r^{1/2}$, $\tau = ip^{1/2}$, and let S be the totality of numbers in the form

$$\Omega = \Delta^{-1/2}(\xi + \eta\rho\tau),$$

where ξ, η are integers such that

(8) $$t\xi \equiv -rs\eta \qquad (\text{mod } \Delta).$$

Let S_1 be the the totality of numbers in the form

$$\Omega_1 = \xi_1\rho + \eta_1\tau,$$

where ξ_1, η_1 are integers. If $U \equiv \Delta^{-1/2}(s\rho + t\tau)$ and

$$\Omega U = \Omega_1,$$

it may be shown that Ω_1 belongs to S_1 if and only if Ω belongs to S. The numbers of S are therefore in one-to-one correspondence with the numbers of S_1.

It may be shown that if μ is an integer and if

$$\Phi_\mu(x, y) \equiv q^{r\mu^2}e^{2r(sx - pty)\mu i}\vartheta_3(\Delta x - rs\mu ki, q^\Delta)\,\vartheta_3(pr\Delta y + prt\mu ki, q^{pr\Delta}),$$

where $k = \log q$, then

$$\Phi_\mu(x, y) = \sum q^{(\xi^2 + pr\eta^2)/\Delta} e^{2(\xi x + pr\eta y)i},$$

where the summation extends over all pairs of integers ξ, η such that

$$\xi \equiv rs\mu, \quad \eta \equiv -t\mu \qquad (\text{mod } \Delta).$$

Then

$$\Phi(x, y) \equiv \sum_{\mu=0}^{\Delta-1}\Phi_\mu(x, y) = \sum_\Omega q^{N(\Omega)}e^{2iR(\Omega X')},$$

where

$$X = \Delta^{1/2}(x + y\rho\tau)$$

and the summation extends over all the numbers of S.

It is readily shown that

$$\Psi(x_1, y_1) \equiv \vartheta_3(rx_1, q^r)\vartheta_3(py_1, q^p) = \sum_{\Omega_1} q^{N(\Omega_1)}e^{2iR(\Omega_1 X_1')},$$

where

$$X_1 = x_1\rho + y_1\tau$$

and the summation extends over all the numbers of S_1.

Let x_1, y_1 be defined in terms of x, y by

$$X_1 = XU.$$

By means of $\Omega_1 = \Omega U$ we have a one-to-one correspondence between the terms of Φ and Ψ. For corresponding terms,

$$N(\Omega_1) = N(\Omega)N(U) = N(\Omega),$$
$$\Omega_1 X_1' = (\Omega U)(U'X') = \Omega X'.$$

Hence corresponding terms are equal and

$$\Phi(x, y) = \Psi(x_1, y_1).$$

If this is expressed in terms of theta functions and y is replaced by $-y/pr$, we obtain Schroeter's formula:

$$\vartheta_3(rsx + ty, q^r)\vartheta_3(-ptx + sy, q^p)$$
$$= \sum_{\mu=0}^{\Delta-1} q^{r\mu^2} e^{2(rsx+ty)\mu i} \vartheta_3(\Delta x - rs\mu ki, q^\Delta)\vartheta_3(\Delta y - prt\mu ki, q^{pr\Delta}).$$

8. **An analogue of Schroeter's identity.** Let p, r, s, t be defined as in the preceding paragraph, except that

$$rs^2 + pt^2 = \Delta^2,$$

where Δ is a positive integer. Let S [S_1] be the totality of numbers in the form

$$\Omega = \xi + \eta\rho\tau \quad [\Omega_1 = \xi_1\rho + \eta_1\tau],$$

where ξ, η [ξ_1, η_1] are integers such that

$$t\xi \equiv -rs\eta \qquad [t\xi_1 \equiv s\eta_1] \qquad\qquad (\text{mod } \Delta).$$

Let $U = s\rho + t\tau$. If

(9) $$\Omega U = \Delta\Omega_1,$$

it may be shown that Ω_1 belongs to S_1 if and only if Ω belongs to S. Let μ be an arbitrarily chosen integer and

$$\Phi_\mu(x, y) \equiv q^{r\mu^2\Delta^2} e^{2r(sx-pty)\mu i}\vartheta_3(\Delta x - rs\mu\Delta ki, q^{\Delta^2})\vartheta_3(pr\Delta y + prt\mu\Delta ki, q^{pr\Delta^2}).$$

Then

$$\Phi_\mu(x, y) = \sum q^{\xi^2+pr\eta^2} e^{2(\xi x + pr\eta y)i},$$

where the summation extends over all pairs of integers ξ, η such that

$$\xi \equiv rs\mu, \qquad \eta \equiv -t\mu \qquad\qquad (\text{mod } \Delta).$$

Then

$$\Phi(x,\,y) \equiv \sum_{\mu=0}^{\Delta-1}\Phi_\mu(x,\,y) = \sum_\Omega q^{N(\Omega)}e^{2iR(\Omega X')},$$

where the summation extends over all the numbers of S and $X = x + y\rho\tau$.

If we set

$$\Psi(x_1,\,y_1) \equiv \sum_{\mu=0}^{\Delta-1}q^{\Delta^2\mu^2}e^{2\mu(rsz_1+pty_1)i}\vartheta_3(r\Delta x_1 - rs\mu\Delta ki,\,q^{r\Delta^2})\vartheta_3(p\Delta y_1 - pt\mu\Delta ki,\,q^{p\Delta^2}),$$

it may be shown in a similar manner that

$$\Psi(x_1,\,y_1) = \sum_{\Omega_1} q^{N(\Omega_1)}e^{2iR(\Omega_1 X_1')},$$

where the summation extends over all the numbers of S_1 and $X_1 = x_1\rho + y_1\tau$.

By (9), the terms of Φ are in one-to-one correspondence with the terms of Ψ; and furthermore, for corresponding terms, $N(\Omega) = N(\Omega_1)$. Hence if we set

$$\Delta X_1 = XU$$

it follows that

$$\Phi(x,\,y) = \Psi(x_1,\,y_1).$$

If we express this in terms of theta functions and replace q^{Δ^2}, Δx, $pr\Delta y$, $r\Delta x_1$, $p\Delta y_1$ by q, x, y, x_1, y_1 respectively, we obtain the identity below.

THEOREM 2. *If p, r are positive integers and s, t, u, v are integers such that*

$$rs^2 + pt^2 = \Delta^2,$$

where Δ is a positive integer prime to t, then

$$\sum_{\mu=0}^{\Delta-1}q^{r\mu^2}e^{2\mu x_1 i}\vartheta_3\left(x - \frac{rs\mu ki}{\Delta},\,q\right)\vartheta_3\left(y + \frac{prt\mu ki}{\Delta},\,q^{pr}\right)$$

$$= \sum_{\mu=0}^{\Delta-1}q^{\mu^2}e^{2\mu xi}\vartheta_3\left(x_1 - \frac{rs\mu ki}{\Delta},\,q^r\right)\vartheta_3\left(y_1 - \frac{pt\mu ki}{\Delta},\,q^p\right),$$

where

$$x_1 = (rsx - ty)/\Delta, \quad y_1 = (ptx + sy)/\Delta.$$

For the case $r = s = t = 1$, $p = 3$, the identity of the above theorem is equivalent to the identity mentioned at the end of §6. For $r = p = 2$, $s = t = 1$, the above identity reduces to

$$\vartheta_3(x,\,q)\vartheta_3(y,\,q) = \vartheta_3(x_1,\,q^2)\vartheta_3(y_1,\,q^2) + \vartheta_2(x_1,\,q^2)\vartheta_2(y_1,\,q^2),$$

$$x_1 = x - y, \quad y_1 = x + y.$$

This is a special case of Schroeter's identity.

9. **An extension of Schroeter's identity.** In this and the next paragraph we shall obtain identities in functions of four variables similar to Schroeter's identity and to the identity of the preceding paragraph.

Let a, b, c be positive integers and let s, t, u, v be integers such that

$$s^2 + bct^2 + cau^2 + abv^2 \equiv \Delta$$

is prime to $\delta \equiv s^2 + bct^2$. Let

$$I = (bc)^{1/2}i, \quad J = (ca)^{1/2}j, \quad K = (ab)^{1/2}k,$$

where i, j, k are the quaternion units, and let

$$A = su - btv, \qquad B = sv + ctu.$$

Let S be the totality of quaternions in the form

$$\Omega = \Delta^{-1/2}(\xi + \eta I + \zeta J + \lambda K),$$

where ξ, η, ζ, λ are integers such that

(10)
$$\delta\xi \equiv acA\zeta + abB\lambda,$$
$$\delta\eta \equiv -aB\zeta + aA\lambda \qquad \text{(mod } \Delta\text{)}.$$

Let S_1 be the totality of quaternions in the form

$$\Omega_1 = \xi_1 + \eta_1 I + \zeta_1 J + \lambda_1 K,$$

where ξ_1, η_1, ζ_1, λ_1 are integers, and let

$$U = \Delta^{-1/2}(s + tI + uJ + vK).$$

It may be shown that if

(11) $$\Omega U = \Omega_1,$$

then Ω_1 belongs to S_1 if and only if Ω belongs to S. We have therefore by means of (11) a one-to-one correspondence between the elements of S and those of S_1.

If ξ, η, ζ, λ satisfy (10), there is a uniquely determined pair of integers μ, ν; $0 \leq \mu < \Delta$, $0 \leq \nu < \Delta$, such that

(12) $$\xi \equiv \delta\mu, \quad \eta \equiv \delta\nu \qquad \text{(mod } \Delta\text{)}.$$

By means of (10) we find

(13) $$\zeta \equiv -A\mu + bB\nu, \quad \lambda \equiv -(B\mu + cA\nu) \qquad \text{(mod } \Delta\text{)}.$$

Conversely, if μ, ν are arbitrarily chosen integers and if ξ, η, ζ, λ satisfy (12) and (13), they also satisfy (10). Let

$$\rho = -A\mu + bB\nu, \quad \tau = -(B\mu + cA\nu),$$

$$\Phi_{\mu,\nu} \equiv q^{\delta(\mu^2 + bc\nu^2)} e^{2(\delta\mu x + bc\delta\nu y + ca\rho z + ab\tau w)i} \vartheta_3(\Delta x - \delta\mu ki, q^\Delta)$$

$$\cdot \vartheta_3[bc(\Delta y - \delta\nu ki), q^{bc\Delta}]\vartheta_3[ca(\Delta z - \rho ki), q^{ca\Delta}]\vartheta_3[ab(\Delta w - \tau ki), q^{ab\Delta}]$$

where $k = \log q$. It may be shown that

$$\Phi_{\mu\nu} = \sum q^{(\xi^2 + bc\eta^2 + ac\zeta^2 + ab\lambda^2)/\Delta} e^{2(\xi x + bc\eta y + ca\zeta z + ab\lambda w)i}$$

where the summation extends over all quadruples ξ, η, ζ, λ which satisfy (12) and (13).

If we set $X = \Delta^{1/2}(x + yI + zJ + wK)$ it follows that

$$\Phi(x, y, z, w) \equiv \sum_{\mu,\nu=0}^{\Delta-1} \Phi_{\mu\nu} = \sum_{\Omega} q^{N(\Omega)} e^{2iR(\Omega X')},$$

where the summation extends over all the elements of S. It is readily shown that if

$$\Psi(x_1, y_1, z_1, w_1) \equiv \vartheta_3(x_1, q)\vartheta_3(bcy_1, q^{bc})\vartheta_3(caz_1, q^{ca})\vartheta_3(abw_1, q^{ab}),$$

then

$$\Psi(x_1, y_1, z_1, w_1) = \sum_{\Omega_1} q^{N(\Omega_1)} e^{2iR(\Omega_1 X_1')},$$

where the summation extends over all the elements of S_1 and $X_1 = x_1 + y_1 I + z_1 J + w_1 K$. By means of (11) there is a one-to-one correspondence between the terms of Φ and those of Ψ. If we let x_1, y_1, z_1, w_1 be defined in terms of x, y, z, w by $X_1 = XU$, it may be shown as in the preceding proofs that the corresponding terms of Φ and Ψ are equal and hence

$$\Phi(x, y, z, w) = \Psi(x_1, y_1, z_1, w_1).$$

Expressing this in terms of theta functions, replacing y, z, w by $y/(bc)$, $z/(ca), w/(ab)$ respectively, and making similar replacements for y_1, z_1, w_1, we obtain the identity below.

THEOREM 3. *Let a, b, c be positive integers and let s, t, u, v be integers such that*

$$s^2 + bct^2 + cau^2 + abv^2 \equiv \Delta$$

is prime to $\delta = s^2 + bct^2$. Let $A = su - btv$, $B = sv + ctu$. Then

$$\vartheta_3(x_1, q)\vartheta_3(y_1, q^{bc})\vartheta_3(z_1, q^{ca})\vartheta_3(w_1, q^{ab}) = \sum_{\mu,\nu=0}^{\Delta-1} q^{\delta(\mu^2 + bc\nu^2)} e^{2(\delta\mu x + \delta\nu y + \rho z + \tau w)i}$$

$$\cdot \vartheta_3(\Delta x - \delta\mu ki, q^\Delta)\vartheta_3(\Delta y - bc\delta\nu ki, q^{bc\Delta})\vartheta_3(\Delta z - ca\rho ki, q^{ca\Delta})\vartheta_3(\Delta w - ab\tau ki, q^{ab\Delta});$$

$$x_1 = sx - ty - uz - vw,$$
$$y_1 = bctx + sy + bvz - cuw,$$
$$z_1 = caux - avy + sz + ctw,$$
$$w_1 = abvx + auy - btz + sw,$$
$$\rho = -A\mu + bB\nu,$$
$$\tau = -B\mu - CA\nu.$$

10. An extension of Theorem 2. Let a, b, c be positive integers and let s, t, u, v be integers such that

$$s^2 + bct^2 + cau^2 + abv^2 = \Delta^2,$$

where Δ is a positive integer prime to $\delta = s^2 + bct^2$. Let A, B, ρ, τ be defined as in the preceding paragraph and let

$$A_1 = su + btv, \quad B_1 = sv - ctu.$$

Let S be the totality of quaternions in the form

$$\Omega = \xi + \eta I + \zeta J + \lambda K,$$

where $\xi, \eta, \zeta, \lambda$ are integers which satisfy (10). Let S_1 be the totality of quaternions in the form

$$\Omega_1 = \xi_1 + \eta_1 I + \zeta_1 J + \lambda_1 K,$$

where $\xi_1, \eta_1, \zeta_1, \lambda_1$ are integers such that

(14)
$$\delta\xi_1 \equiv -acA_1\zeta_1 - abB_1\lambda_1,$$
$$\delta\eta_1 \equiv aB_1\zeta_1 - aA_1\lambda_1 \qquad (\bmod \Delta).$$

As in the preceding paragraph, if $\xi, \eta, \zeta, \lambda$ satisfy (10), there is a uniquely determined pair of integers μ, ν; $0 \le \mu < \Delta$, $0 \le \nu < \Delta$, such that

(15) $$\xi \equiv \delta\mu, \quad \eta \equiv \delta\nu, \quad \zeta \equiv \rho, \quad \eta \equiv \tau \qquad (\bmod \Delta).$$

If we set

$$\Phi_{\mu\nu} \equiv q^{\delta(\mu^2 + bc\nu^2)\Delta^2} e^{2(\delta\mu x + \delta bc\nu y + ca\rho z + ab\tau w)i} \vartheta_3(\Delta x - \delta\mu\Delta ki, q^{\Delta^2})$$

$$\cdot \vartheta_3(bc\Delta y - \delta\nu bc\Delta ki, q^{bc\Delta^2}) \vartheta_3(ca\Delta z - \rho ca\Delta ki, q^{ca\Delta^2}) \vartheta_3(ab\Delta w - \tau ab\Delta ki, q^{ab\Delta^2}),$$

it may be shown that

$$\Phi_{\mu\nu} = \sum q^{\xi^2 + bc\eta^2 + ca\zeta^2 + ab\lambda^2} e^{2(\xi x + bc\eta y + ca\zeta z + ab\lambda w)i},$$

where the summation extends over all quadruples which satisfy (15). Hence

$$\Phi(x, y, z, w) \equiv \sum_{\mu, \nu=0}^{\Delta-1} \Phi_{\mu\nu} = \sum_{\Omega} q^{N(\Omega)} e^{2iR(\Omega X')},$$

where the summation extends over all the elements of S and $X = x + I + zJ + wK$.

Set

$$\rho_1 = A_1\mu - bB_1\nu, \quad \tau_1 = B_1\mu + cA_1\nu.$$

In $\Phi_{\mu\nu}$ replace ρ, τ by ρ_1, τ_1 respectively and let $\Psi_{\mu\nu}$ be the resulting function. It may be shown as for Φ that

$$\Psi(x_1, y_1, z_1, w_1) = \sum_{\mu, \nu=0}^{\Delta-1} \Psi_{\mu\nu} = \sum_{\Omega_1} q^{N(\Omega_1)} e^{2iR(\Omega_1 X_1')},$$

where the summation extends over all the elements of S_1 and $X_1 = x_1 + y_1 I + z_1 J + w_1 K$. If we set $\Delta X_1 = XU$, $\Delta\Omega_1 = \Omega U$ where $U = s + tI + uJ + wK$, it may be shown that Ω_1 belongs to S_1 if and only if Ω belongs to S and that the corresponding terms of Φ and Ψ are equal. Hence

$$\Phi(x, y, z, w) = \Psi(x_1, y_1, z_1, w_1).$$

If we express this in terms of theta functions, replace q^{Δ^2}, Δx, Δbcy, Δcaz, Δabw by q, x, y, z, w respectively and make similar replacements for x_1, y_1, z_1, w_1, we obtain the identity below.

THEOREM 4. *Let a, b, c be positive integers and let s, t, u, v be integers such that*

$$s^2 + bct^2 + cau^2 + abv^2 = \Delta^2,$$

where Δ is a positive integer prime to $\delta \equiv s^2 + bct^2$. Let

$$A = su - btv, \quad A_1 = su + btv, \quad \rho = -A\mu + bB\nu, \quad \rho_1 = A_1\mu - bB_1\nu,$$
$$B = sv + ctu, \quad B_1 = sv - ctu, \quad \tau = -B\mu - cA\nu, \quad \tau_1 = B_1\mu + cA_1\nu,$$

where μ, ν are arbitrarily chosen integers. Let

$$F(x, y, z, w) \equiv \sum_{\mu, \nu=0}^{\Delta-1} q^{\delta(\mu^2 + bc\nu^2)} e^{2(\delta\mu x + \delta\nu y + \rho z + \tau w)i/\Delta} \vartheta_3\left(x - \frac{\delta\mu ki}{\Delta}, q\right)$$
$$\cdot \vartheta_3\left(y - \frac{\delta bc\nu ki}{\Delta}, q^{bc}\right) \vartheta_3\left(z - \frac{\rho ca ki}{\Delta}, q^{ca}\right) \vartheta_3\left(w - \frac{\tau ab ki}{\Delta}, q^{ab}\right),$$

and let F_1 be the function obtained from F by replacing ρ, τ by ρ_1, τ_1 respectively. Then

$$F(x, y, z, w) = F_1(x_1, y_1, z_1, w_1)$$

where

$$x_1 = (sx - ty - uz - vw)/\Delta, \qquad z_1 = (caux - avy + sz + ctw)/\Delta,$$
$$y_1 = (bctx + sy + bvz - cuw)/\Delta, \quad w_1 = (abvx + auy - btz + sw)/\Delta.$$

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